Preferable-Interval Method for the Multiple-Sink Fixed-Charge Transportation Problem

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1 Introduction

In this technical report, we study the *multiple-sink fixed-charge transportation problem* (MSFCTP), which arises frequently in application areas of scheduling and cost control, such as facility planning, capital budgeting, resource allocation, buffer allocation, pollution control, etc. The MSFCTP problem involves the distribution from a set of supply centers (sources) to a set of demand centers (destinations) such that the demand at each destination is satisfied without exceeding the supply at any source. The objective is to determine a distribution scheme that has the least cost of transformation.

The MSFCTP is traditionally formulated as a mixed integer programming problem described as follows [23]:

$$\begin{aligned} \min \quad & TC = \sum_{i=1}^{m} \sum_{k=1}^{n} (c_{ik}x_{ik} + f_{ik}y_{ik}) \\ \text{s.t.} \quad & \sum_{k=1}^{n} x_{ik} = S_i \quad \text{for} \quad 1 \le i \le m, \\ & \sum_{i=1}^{m} x_{ik} = D_k \quad \text{for} \quad 1 \le k \le n, \\ & 0 \le x_{ik} \le m_{ik}y_{ik} \quad \text{for} \quad 1 \le i \le m, 1 \le k \le n, \\ & y_{ik} \in \{0,1\} \quad \text{for} \quad 1 \le i \le m, 1 \le k \le n, \end{aligned} \tag{1.1}$$

where n is the number of destinations, m is the number of sources, c_{ik} is the cost per unit amount transported from source i to destination k, x_{ik} is the amount transported from source i to destination k, $f_{ik} \ge 0$ is the fixed-charge incurred if $x_{ik} \ne 0$, $S_i > 0$ is the supply available at source i, $D_k > 0$ is the demand at destination k, $m_{ik} = \min\{S_i, D_k\}$ is the maximum amount that can be transported from source i to destination k, and y_{ik} is 1 if $x_{ik} \ne 0$ and 0 otherwise. We assume that $\sum_i S_i = \sum_k D_k$.

The *fixed-charge problem* (FCP) was first formulated by Hirsch and Dantzig in 1954 [17]. In 1961, Balinski showed that the *fixed-charge transportation problem* (FCTP) is a special case of the FCP and presented an approximation solution [5]. Initial approximation solutions to the problem were mainly heuristic [5, 9, 24, 26, 28]. In 1968, Murty developed the first exact solution to the FCTP [22]. As he pointed out, the method is most useful when the fixed charges are quite small compared to the transportation cost. Other exact approaches specialized to the problem include [11, 13, 14, 26, 27]. Kennington and Unger proposed a branch-and-bound procedure for the FCTP [18]. The bounds have been improved by Barr et al. [6], Cabot and Erenguc [8], Schaffer and O'Leary [25], Palekar et al. [23], Haberl et al. [15], etc. Due to their application potential and computational challenge, these problems continue to be the focus of considerable research [1, 2, 3, 7, 10, 12]. Note that when $n = 1$, the above interested problem becomes a *single-sink fixed-charge transportation problem* (SSFCTP) [4, 16, 19, 20, 21].

Recently we have developed a preferable intervals solution approach for SSFCTPs [20, 21]. Several heuristic algorithms based on preferable intervals of polynomial time complexity are provided. We have compared our heuristics with CPLEX, which is a benchmark commercial software widely used in both academic and industrial communities. The computation shows that our method is fast and efficient, which works very well for problems of both large and small scales. Encouraged by the previous results, we will extend the preferable interval method for the MSFCTP.

The rest of this report is organized as follows. We start with definition and properties of preferable intervals in Section 2. Section 3 is devoted to properties of the optimal solutions to the MSFCTP. In Section 4, heuristic algorithms are presented followed by computational experience and our summary in Sections 5 and 6, respectively.

2 Preferable Intervals

Standard interval notations are used in this paper. In particular, $[a, a] = \{a\}$ and $(a, a] = \emptyset$ for all real number a. If $a > b$, then $[a, b] = \emptyset$ and $(a, b] = \emptyset$. Here $\emptyset$ stands for the empty set.

There are three different ways to assign work load of a destination to two sources. One may choose only one of the two sources, or both sources. We want to determine when the employment of only one source will yield the least cost among the three cases. Motivated by this purpose, we define the preferable interval of one source against another source for a certain destination as the following.

Definition 2.1 *For destination k, the preferable interval of source i against source j is defined as*

$$ {}_kI_j^i = \begin{cases} {}_kA_j^i \cup {}_kB_j^i & \text{if sources } i \text{ and } j \text{ are distinct} \\ [0, m_{ik}] & \text{if sources } i \text{ and } j \text{ are identical} \end{cases} $$

where

$$ {}_kA_j^i = \{x \in [0, m_{ik}] : f_{ik} + c_{ik}x < f_{jk} + c_{jk}x\}, $$

$$ \begin{aligned} {}_kB_j^i &= (m_{jk}, m_{ik} \cdot \operatorname{sgn}(f_{jk} + (c_{jk} - c_{ik})m_{jk})] \\ &= \begin{cases} (m_{jk}, m_{ik}] & \text{if } m_{jk} < m_{ik} \text{ and } f_{jk} + (c_{jk} - c_{ik})m_{jk} > 0, \\ \emptyset & \text{otherwise,} \end{cases} \end{aligned} $$

and $\operatorname{sgn}(x)$ *is the signum function of* x.

It is clear that ${}_kI_j^i \subseteq [0, m_{ik}]$ for any $1 \leq i, j \leq M$. Next, we discuss the implications of the preferable interval ${}_kI_j^i$ for different sources i and j. We have the following property of preferable intervals.

Proposition 2.2 *Consider different sources i and j for destination k. If the demand magnitude $x \leq m_{ik}$, then sole source i is preferred to source j or any combination of both sources i and j in terms of cost if and only if $x \in {}_kI_j^i$.*

Proof. Assume that $x \in {}_kA_j^i \neq \emptyset$. By the definition of ${}_kA_j^i$, it is clear that

$$ f_{ik} + c_{ik}x < f_{jk} + c_{jk}x. \tag{2.1} $$

Moreover, it can be proved that

$$ f_{ik} + c_{ik}x < f_{jk} + c_{jk}\xi + f_{ik} + c_{ik}(x - \xi) \tag{2.2} $$

or, equivalently,

$$ 0 < f_{jk} + (c_{jk} - c_{ik})\xi \tag{2.3} $$

for $0 < \xi < x$. In order to show (2.3), consider the following two cases. If $c_{ik} < c_{jk}$, since $f_{ik} \geq 0$, then (2.3) is trivially true. If $c_{jk} \leq c_{ik}$, then (2.1) and $\xi < x$ imply

$$ 0 < f_{ik} < f_{jk} + (c_{jk} - c_{ik})x \leq f_{jk} + (c_{jk} - c_{ik})\xi $$

which is exactly (2.3). Hence (2.3) is always true when $0 < \xi < x$, and so is (2.2). Inequality (2.1) indicates that source i yields less cost than source j. Inequality (2.2) implies that using source i leads less cost than using both sources. Therefore, when $x \in_k A_j^i$, an employment of solely source i will yield the least cost among all possible assignments indicated above.

Secondly, suppose that $x \in {}_kB_j^i \neq \emptyset$. This implies that $m_{jk} < m_{ik}$ and

$$ 0 < f_{jk} + (c_{jk} - c_{ik})m_{jk}. \tag{2.4} $$

Since $m_{jk} < x$, the shipment cannot be made by source j itself. So we need only compare the cost by sole source i and the cost by both sources i and j; i.e. we need to ensure (2.2) for $0 < \xi \leq m_{jk}$. By similar arguments as the proof of (2.3), one can conclude that condition

(2.4) will make (2.2) valid. Therefore, when $x \in {}_kB^i_j$, using only source i will again yield the least costly solution among the other possible assignments to both sources.

On the other hand, when $m_{jk} < x$, in order to guarantee that the employment of sole source i yields the least cost, we need to check the following two cases. Case (i) If $x \leq m_{jk}$, then (2.1) must hold, and (2.2) must hold for $0 < \xi < x$. Notice that (2.1) implies (2.2) in this situation, which leads to the fact that $x \in_k A^i_j$ and hence $x \in {}_kI^i_j$. Case (ii) If $x > m_{jk}$, then only (2.2) needs to be valid for $0 < \xi \leq m_{jk}$. If $c_{ik} < c_{jk}$, then (2.2) is clearly true for any $0 < \xi \leq m_{jk}$; in particular, it holds when $\xi = m_{jk}$, which is actually (2.4). If $c_{jk} \leq c_{ik}$, then (2.2) holds for any $0 < \xi \leq m_{jk}$ if and only if (2.4) is satisfied. Both situations give $x \in {}_kB^i_j$ and thus $x \in {}_kI^i_j$. Therefore, one may conclude from the two cases that x must belong to ${}_kI^i_j$ to ensure the least cost by using only source i among all assignments to sources i and j. □

3 Properties of the Optimal Solutions

The MSFCTP is an NP-hard problem. In this section, we present properties of optimal solutions. The properties will be used in the development of the algorithms in Section 4.

Proposition 3.1 *There exists an optimal solution to the MSFCTP (1.1) such that, either for every destination k or for every source i, there is at most one source such that* $0 < x_{ik} < m_{ik}$*. That is, for the entire optimal solution double array* x_{ik}*, either at most one entry in each row is not fully loaded, or at most one entry in each column is not fully loaded.*

Proof. Suppose that, in an optimal solution, there are entries in row k with $0 < x_{ik} < m_{ik}$ and $0 < x_{jk} < m_{jk}$ and entries in another row l with $x_{il} < m_{il}$, $x_{jl} < m_{jl}$, and one of x_{il} and x_{jl} is greater than 0. This is the only case contradicts the statement.

We next modify the solution double array such that the statement of the proposition is satisfied and the total cost is not increased. In particular, for destination k, we can pass a units from source j to source i. (Here a could be negative, in which case units are passed from source j to source i.) To fulfill the subject constraints of (1.1), the two corresponding entries in row l should be adjusted. The new assignments of the four entries are thus $x_{ik}+a$, $x_{jk}-a$, $x_{il}-a$, and $x_{jl}+a$. The change in total cost will be at least

$$a(c_{ik} - c_{jk} - c_{il} + c_{jl}).$$

We will pick a positive if $c_{ik}-c_{jk}-c_{il}+c_{jl} \geq 0$ and pick a negative if $c_{ik}-c_{jk}-c_{il}+c_{jl} < 0$. The total cost will thus not be increased. Moreover, a can be properly chosen such that one of four entries is fully loaded (i.e., e.g., $x_{ik} = m_{ik}$), and another entry in either its same row or column is 0. Therefore, at most one of the four entries are partially loaded, and a reduction of f_{ik}, f_{jk}, f_{il}, or f_{jl} will occur in the total cost.

Doing the same procedure for all groups of entries that violate the statement of the proposition. The new solution will have same or smaller total cost. The proposition thus follows. □

Remark 3.1 The two cases of Proposition 3.1 sometimes cannot both hold. For example, when $S_i > D_k$ for all $1 \leq i \leq m$ and $1 \leq k \leq n$, it could be impossible to find a solution such that at most one entry in every row is partially loaded. But an optimal solution exists with at most one entry in every column is partially loaded in this case. Note that these two cases, in the sense of double array, are the same. Therefore, we may from now on assume that there exists an optimal solution to the MSFCTP (1.1), in which, for every destination k, there is at most one source i such that $0 < x_{ik} < m_{ik}$. The other case can be studied in a similarly way.

If $x_{ik} \neq 0$, then the average cost per unit amount transported from source i to destination k is $e_{ik} = c_{ik} + f_{ik}/x_{ik}$. It is easy to see that when the amount transported increases, the average cost per unit will decrease. In particular, the least possible average cost would be $\epsilon_{ik} = c_{ik} + f_{ik}/m_{ik}$. Clearly, $0 < e_{ik} \leq \epsilon_{ik}$. Therefore, in principle, an optimal solution would include least possible partially loaded sources.

Proposition 3.2 *Consider an optimal solution to the MSFCTP (1.1). Assume that for every destination k, there is at most one source i such that $0 < x_{ik} < m_{ik}$. Then $x_{ik}+m_{jk} \notin {}_kI_j^i$ for all j with $x_{jk} = m_{jk}$.*

Proof. If $x_{ik}+m_{jk} \in {}_kI_j^i$ for some j with $x_{jk} = m_{jk}$, then $x_{ik}+m_{jk} \leq m_{ik}$. The total cost can be reduced by reassigning all units of source j to source i, according to Proposition 2.2. But this contradicts the assumption of optimal solution. □

Remark 3.2 Proposition 3.2 gives a necessary condition of an optimal solution for employing a source. That is, a fully loaded source needs to satisfy Proposition 3.2.

4 Solution Heuristics

In this section, we present several heuristic approaches. The schemes are developed based on properties studied in Section 3. Here it is assumed that for every destination k, there is at most one source i such that $0 < x_{ik} < m_{ik}$.

Algorithm 4.1 (Greedy Heuristic)

Step 1 *Calculate ϵ_{ik} for all entries, save the results in double array E.*

Step 2 *Find the smallest entry in E and fully load the corresponding entry in the double array of x_{ik}. Update E by deleting the row and column of the entry. Repeat Step 2 till E has no entry.*

Step 3 *Using the constraints $\sum_k x_{ik} = S_i$ and $\sum_i x_{ik} = D_k$ to identify the sources and destinations that have not been exhausted. If there is no such source and destination, then stop. Otherwise, put the corresponding ϵ_{ik} together, save the double array as E, and go to Step 2.*

Algorithm 4.2

Step 1 *Find the result of Algorithm* 4.1*.*

Step 2 *For each column, find the entry such that* $0 < x_{ik} < m_{ik}$*. If there is an entry* x_{jk} *does not satisfy Proposition* 3.2*, then passing units from source* j *to* i*. Adjust the other columns according to the constraints* $\sum_k x_{ik} = S_i$ *and* $\sum_i x_{ik} = D_k$*.*

Step 3 *If the total cost is not reduced, then stop. Otherwise, go to Step* 2*.*

5 Computational Experiences

Example 5.1 Consider the MSFCTP given by the following data [5]:

$$C = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 1 & 4 & 3 \\ 4 & 5 & 2 \end{pmatrix}, \quad F = \begin{pmatrix} 10 & 30 & 20 \\ 10 & 30 & 20 \\ 10 & 30 & 20 \\ 10 & 30 & 20 \end{pmatrix},$$

$$D = (20\ 50\ 30), \quad S = (10\ 30\ 40\ 20),$$

where the entries of C, F, D, and S are c_{ik}, f_{ik}, D_k, and S_i, respectively. The double arrays for m_{ik} and ϵ_{ik} are

$$M = \begin{pmatrix} 10 & 10 & 10 \\ 20 & 30 & 30 \\ 20 & 40 & 30 \\ 20 & 20 & 20 \end{pmatrix}, \quad E = \begin{pmatrix} 3 & 6 & 6 \\ 3.5 & 3 & 1.67 \\ 1.5 & 4.75 & 3.67 \\ 4.5 & 6.5 & 3 \end{pmatrix}.$$

By Algorithm 4.1, the solution is

$$X = \begin{pmatrix} 0 & 10 & 0 \\ 0 & 0 & 30 \\ 20 & 20 & 0 \\ 0 & 20 & 0 \end{pmatrix},$$

which has total cost $TC = 380$. Note that the entries x_{32} and x_{42} do not satisfy the Proposition 3.2. We need to adjust the solution. Then by Algorithm 4.2, the solution is

$$X = \begin{pmatrix} 0 & 10 & 0 \\ 0 & 30 & 0 \\ 20 & 10 & 10 \\ 0 & 0 & 20 \end{pmatrix}.$$

The total cost is $TC = 360$, which is the same as the cost in [5].

Example 5.2 Consider the MSFCTP given by the following data [5]:

$$C = \begin{pmatrix} .69 & .64 & .71 & .79 & 1.70 & 2.83 & 2.02 & 5.64 & 5.94 & 5.94 & 5.94 & 7.6 \\ 1.01 & .75 & .88 & .59 & 1.5 & 2.63 & 2.26 & 5.64 & 5.85 & 5.62 & 5.85 & 4.54 \\ 1.05 & 1.06 & 1.08 & .64 & 1.22 & 2.37 & 1.66 & 5.64 & 5.91 & 5.62 & 5.91 & 4.54 \\ 1.94 & 1.50 & 1.56 & 1.22 & 1.98 & 1.98 & 1.36 & 6.99 & 6.99 & 6.99 & 6.99 & 3.68 \\ 1.61 & 1.40 & 1.61 & 1.33 & 1.68 & 2.83 & 1.54 & 4.26 & 4.26 & 4.26 & 4.26 & 2.99 \\ 5.29 & 5.94 & 6.08 & 5.29 & 5.96 & 6.77 & 5.08 & 0.31 & 0.21 & 0.17 & 0.31 & 1.53 \\ 5.29 & 5.94 & 6.08 & 5.29 & 5.96 & 6.77 & 5.08 & 0.55 & 0.35 & 0.40 & 0.19 & 1.53 \\ 5.29 & 6.08 & 6.08 & 5.29 & 5.96 & 6.45 & 5.08 & 2.43 & 2.30 & 2.33 & 1.81 & 2.50 \end{pmatrix},$$

$$F = \begin{pmatrix} 11 & 16 & 18 & 17 & 10 & 20 & 17 & 13 & 15 & 12 & 14 & 14 \\ 14 & 17 & 17 & 13 & 15 & 13 & 16 & 11 & 20 & 11 & 15 & 10 \\ 12 & 13 & 20 & 17 & 13 & 15 & 16 & 13 & 12 & 13 & 10 & 18 \\ 16 & 19 & 16 & 11 & 15 & 12 & 18 & 12 & 18 & 13 & 13 & 14 \\ 19 & 18 & 15 & 16 & 12 & 14 & 20 & 19 & 11 & 17 & 16 & 18 \\ 13 & 20 & 20 & 17 & 15 & 12 & 14 & 11 & 12 & 19 & 15 & 16 \\ 11 & 12 & 15 & 10 & 17 & 11 & 11 & 16 & 10 & 18 & 17 & 12 \\ 17 & 10 & 20 & 12 & 17 & 20 & 16 & 15 & 10 & 12 & 16 & 18 \end{pmatrix},$$

$$D = (20\ 15\ 20\ 15\ 5\ 20\ 30\ 10\ 35\ 25\ 10\ 5), \quad S = (15\ 20\ 45\ 35\ 25\ 35\ 10\ 25),$$

where the entries of C, F, D, and S are c_{ik}, f_{ik}, D_k, and S_i, respectively. The double arrays for m_{ik} and ϵ_{ik} are

$$M = \begin{pmatrix} 15 & 15 & 15 & 15 & 5 & 15 & 15 & 10 & 15 & 15 & 10 & 5 \\ 20 & 15 & 20 & 15 & 5 & 20 & 20 & 10 & 20 & 20 & 10 & 5 \\ 20 & 15 & 20 & 15 & 5 & 20 & 30 & 10 & 35 & 25 & 10 & 5 \\ 20 & 15 & 20 & 15 & 5 & 20 & 30 & 10 & 35 & 25 & 10 & 5 \\ 20 & 15 & 20 & 15 & 5 & 20 & 25 & 10 & 25 & 25 & 10 & 5 \\ 20 & 15 & 20 & 15 & 5 & 20 & 30 & 10 & 35 & 25 & 10 & 5 \\ 10 & 10 & 10 & 10 & 5 & 10 & 10 & 10 & 10 & 10 & 10 & 5 \\ 20 & 15 & 20 & 15 & 5 & 20 & 25 & 10 & 25 & 25 & 10 & 5 \end{pmatrix},$$

$$E = \begin{pmatrix} 1.42 & 1.71 & 1.91 & 1.92 & 3.70 & 4.16 & 3.15 & 6.94 & 6.94 & 6.74 & 7.34 & 10.40 \\ 1.71 & 1.88 & 1.73 & 1.46 & 4.50 & 3.28 & 3.06 & 6.74 & 6.85 & 6.17 & 7.35 & 6.54 \\ 1.65 & 1.93 & 2.08 & 1.77 & 3.82 & 3.12 & 2.19 & 6.94 & 6.25 & 6.14 & 6.91 & 8.14 \\ 2.74 & 2.77 & 2.36 & 1.95 & 4.98 & 2.58 & 1.96 & 8.19 & 7.50 & 7.51 & 8.29 & 6.48 \\ 2.56 & 2.60 & 2.36 & 2.40 & 4.08 & 3.53 & 2.34 & 6.16 & 4.70 & 4.94 & 5.86 & 6.59 \\ 5.94 & 7.27 & 7.08 & 6.42 & 8.96 & 7.37 & 5.55 & 1.41 & 0.55 & 0.93 & 1.81 & 4.73 \\ 6.39 & 7.14 & 7.58 & 6.29 & 9.36 & 7.87 & 6.18 & 2.15 & 1.35 & 2.20 & 1.89 & 3.93 \\ 6.14 & 6.75 & 7.08 & 6.09 & 9.36 & 7.45 & 5.72 & 3.93 & 2.70 & 2.81 & 3.41 & 6.10 \end{pmatrix}.$$

By Algorithm 4.2, we finally get the solution

$$X = \begin{pmatrix} 15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 15 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 \\ 5 & 15 & 0 & 0 & 5 & 20 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 30 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 20 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 35 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 25 & 0 & 0 \end{pmatrix}.$$

The total cost is $TC = 497.25$, which is better than the cost 497.55 in [5].

6 Summary and Conclusions

In this paper, we extended the concept of preferable intervals to the MSFCTP. The effectiveness of this approach was illustrated by two computational examples. The procedure is innovative and powerful. It is our belief that the techniques could be improved and utilized for branch-and-bound method, which is an on-going project.

References

[1] V. Adlakha and K. Kowalski, On the fixed-charge transportation problem, Omega 27 (1999), pp. 381-388.

[2] V. Adlakha and K. Kowalski, A simple heuristic for solving small fixed-charge transportation problems, Omega 31 (2003), pp. 205-211.

[3] V. Adlakha, K. Kowalski, R. R. Vemuganti, and B. Lev, More-for-less algorithm for fixed-charge transportation problems, Omega 35 (2007), 116.

[4] B. Alidaee and G. A. Kochenberger, A note on a simple dynamic programming approach to the single-sink, fixed-charge transportation problem, Transportation Science 39 (2005), pp. 140-143.

[5] M. L. Balinski, Fixed-cost transportation problems, Naval Research Logistics Quarterly 8 (1961), pp. 41-54.

[6] R. S. Barr, F. Glover, and D. Klingman, A new optimization method for large scale fixed charge transportation problems, Operations Research 29 (1981), 448-463.

[7] W. L. Bruce and A. W. Chris, Revised-modified penalties for fixed charge transporation problems, Management Science 43 (1997), 1431.

[8] V.A. Cabot and S.S. Erenguc, Some branch-and-bound procedures for fixed-cost transportation problems, Naval Res. Logist. Quart., 31 (1984), 145-154.

[9] L. Cooper and C. Drebes, An approximation solution method fo the fixed charge problem, Naval Research Logistics Quarterly 14 (1967), pp. 101-103.

[10] M. Diaby, Successive linear approximation procedure for generalized fixed-charge transportation problems, The Journal of the Operational Research Society 42 (1991), 991.

[11] R.S. Frank, On the fixed charge hitchcock transportation problem, unpublished dissertation. The John Hopkins University, MD, 1972.

[12] M. Gothe-Lundgren and T. Larsson, A set covering reformulation of the pure fixed charge transportation problem, Discrete Applied Mathematics 48 (1994), pp. 245-259.

[13] P. Gray, Exact solution of the fixed-charge transportation problem, Operations Research 19 (1971), pp. 1529-1538.

[14] J. Haberl, Exact algorithm for solving a special fixed charge linear programming problem, Journal of Optimization Theory and Applications 69 (1991), pp. 489-529.

[15] J. Haberl, C. Nowak, H. Stettner, G. Stoiser, and H. Woschits, A branch-and-bound algorithm for solving a fixed charge problem in the profit optimization of sawn timber production., ZOR - Methods and Models of Operations Research 35 (1991), pp. 151-166.

[16] Y. T. Herer, M. J. Rosenblatt, and I. Hefter, Fast algorithms for single-sink fixed charge transportation problems with applications to manufacturing and transportation, Transportation Science 30 (1996), pp. 276-290.

[17] W. Hirsch and G. B. Dantzig, The fixed charge problem, Naval Research Logistics Quarterly 15 (1968), pp. 413-424.

[18] J. Kennington and V.E. Unger, A new branch-and-bound algorithm for the fixed charge transportation problem, Management Science 22 (1976), 1116-1126.

[19] A. Klose, Algorithms for solving the single-sink fixed-charge transportation problem, Computers and Operations Research 35 (2008), pp. 2079-2092.

[20] R. Lin, Preferable-interval approaches to the single-sink fixed-charge transportation problem, Texas A&M International University Texas Center Border Research Reports, Vol. 7, (2006).

[21] R. Lin, H. Wang, B. Alidaee, G. Kochenberger, and Y. Xu, Simple, fast heuristics based on preferable intervals for the single-sink, fixed-charge transportation problem, Submitted to Naval Research Logistics.

[22] K.G. Murty, Solving the fixed charge transportation problem by extreme point ranking, Operations Research 16 (1968), pp. 268-279.

[23] U.S. Palekar, M.H. Karwan, and S. Zionts, A branch-and-bound method for the fixed charge transportation problem, Management Science 36 (1990), pp. 1092-1105.

[24] P. Robers and L. Cooper, A study of the fixed charge transportation problem, Computers & Mathematics with Applications 2 (1976), pp. 125-135.

[25] J. R. Schaffer and D. E. O'Leary, Use of penalties in a branch and bound procedure for the fixed charge transportation problem, European Journal of Operational Research 43 (1989), pp. 305-312.

[26] D.I. Steinberg, The fixed charge problem, Naval Research Logistics Quarterly 17 (1970), pp. 217-236.

[27] C.J. Tompkins, Group theoretic structures in the fixed charge transportation problem, unpublished dissertation, Georgia Institute of Technology, 1971.

[28] W.E. Walker, An adjacent extreme point algorithm for the fixed charge problem, Management Science 22 (1976), pp. 587-596.